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## On geometric polygroups

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### Abstract

In this paper, we introduce a geodesic metric space called generalized Cayley graph ( $gCay(P,S)$ ) on a finitely generated polygroup. We define a hyperaction of polygroup on  $gCay$  graph and give some properties of this hyperaction. We show that  $gCay$  graphs of a polygroup by two different generators are quasi-isometric. Finally, we express a connection between finitely generated polygroups and geodesic metric spaces.

### 1 Introduction

Geometric group theory is a field in mathematics devoted to the study of finitely generated groups via exploring the connections between algebraic properties of such groups and topological and geometric properties of spaces on which these groups act. An idea in geometric group theory is to consider finitely generated groups themselves as geometric objects. This is usually done by studying the Cayley graphs of groups which, in addition to the graph structure, are endowed with the structure of a metric space, given by the so-called word metric. Geometric group theory, as a distinct area, is relatively new and became a clearly identifiable branch of mathematics in the late 1980s. Geometric group theory closely interacts with low-dimensional topology, hyperbolic geometry, algebraic topology [1, 2], computational group theory and differential geometry. There are also substantial connections with complexity theory, mathematical logic, the study of Lie Groups and their discrete

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subgroups, dynamical systems, probability theory, K-theory, and other areas of mathematics. The reader will find in [16, 25] some basic definitions and theorems about geometric group theory. In 1934 Marty at 8th congress of Scandinavian Mathematicians introduced the notion of hypergroup as a generalization of groups and after, he proved its utility in solving some problems of groups, algebraic functions and rational fractions [29]. Surveys of the theory can be found in the books of Corsini [11], Davvaz and Leoreanu-Fotea [14], Davvaz and Cristea [15], Corsini and Leoreanu [12], Vougiouklis [34] and in the paper of Hoskova and Chvalina [19]. In recent years, the theory of hyperstructures has been refreshed in connection with various fields. This is basically done by A. Connes and C. Consani in connection to number theory, incidence geometry, and geometry in characteristic one [7, 8, 9], O.Viro in connection to tropical geometry [32, 33], and M. Marshall in connection to quadratic forms and real algebraic geometry [18, 28]. Moreover, hyperstructures have certain relations with recently introduced algebraic objects such as supertropical algebras by Z. Izhakian and L. Rowen [20, 21], blueprints by O. Lorscheid [26, 27]. These are algebraic objects which aim to provide a firm algebraic foundation to tropical geometry. J. Jun also applied an idea of hyperstructures to generalize the definition of valuations in [24]. Some other connections of algebraic geometry over hyperstructures can be find in [3, 6, 17, 22, 23, 30].

In this paper first, using some geometric notions we generalize the notion of Cayley graph for the class of finitely polygroups [4]. Second, we investigate some properties of metric  $d_S$  on the generalized Cayley graph  $gCay(P, S)$ . Third by the notion of hyperaction of hypergroups introduced in [36] we show that a left hyperaction of a good polygroup  $P = \langle S \rangle$  on  $gCay(P, S)$  is proper and strongly cobounded. Then we prove that  $gCay(P, S)$  is well-defined up to quasi-isometry. Finally, we express the definition of geodesic metric space and then we show that  $gCay(P, S)$  is a geodesic metric space. Also we prove that if polygroup  $P$  that for all  $a, b \in P$ ,  $|ab| < \infty$  acts properly and strongly coboundedly on the geodesic metric space  $X$ , then  $P$  is finitely generated and  $gCay(P, S)$  is quasi-hyperisometric to  $X$ . In the following we recall some basic notions of hypergroup theory.

## 2 Preliminaries

**Definition 2.1.** ([5],[11], [12]) *Let  $H$  be a non-empty set and  $*$ :  $H \times H \rightarrow \mathcal{P}^*(H)$  be a hyperoperation. The couple  $(H, *)$  is called a hypergroupoid. For any two non-empty subset  $A$  and  $B$  of  $H$  and  $x \in H$ , we define*

$$A * B = \bigcup_{a \in A, b \in B} a * b, \quad A * x = A * \{x\}.$$

**Definition 2.2.** ([11], [12], [13]) A hypergroupoid  $(H, *)$  is called hypergroup if for all  $a, b, c$  of  $H$ , it satisfies the following conditions:

(1)  $(a * b) * c = a * (b * c)$ , which means that

$$\bigcup_{u \in a * b} u * c = \bigcup_{v \in b * c} a * v,$$

(2)  $a * H = H = H * a$ .

**Definition 2.3.** ([11], [12]) Let  $(H, *)$  be a hypergroup and  $\emptyset \neq K \subset H$ . We say that  $(K, *)$  is a subhypergroup of  $H$  if for all  $x \in K$  we have  $K * x = K = x * K$ .

Let  $(H, *)$  be a hypergroup, an element  $e$  of  $H$  is called *identity* if for all  $x \in H$ ,  $x \in x * e \cap e * x$  and for  $a \in H$  an element  $a'$  of  $H$  is called an *inverse* of  $a$ , if  $e \in a' * a \cap a * a'$ , for some identity  $e$ . We denote the set of identities of  $H$  by  $E(H)$  and the set of inverses of  $a$  by  $i(a)$ .

For any  $a, b \in (H, *)$ , we define  $a/b = \{x \mid a \in x * b\}$  and  $a \setminus b = \{y \mid b \in a * y\}$ . Now let  $A$  be a non-empty subset of hypergroup  $(H, *)$ . Denote  $A_0 = A \cup (A * A) \cup (A/A) \cup (A \setminus A)$  and  $A_{n+1} = A_n \cup (A_n * A_n) \cup (A_n/A_n) \cup (A_n \setminus A_n)$ , where  $A/B = \cup_{a \in A, b \in B} a/b$  and  $A \setminus B = \cup_{a \in A, b \in B} a \setminus b$ .

**Theorem 2.4.** ([11], Theorem 78)  $\langle A \rangle = \cup_{n \geq 0} A_n$  is the least closed subhypergroup of  $H$ .

Let  $H = \langle A \rangle$  and  $A$  be a finite set ( $|A| < \infty$ ). Then we say that the hypergroup  $H$  is a finitely generated hypergroup.

**Definition 2.5.** ([4], [10], [35]) A polygroup  $(P, \cdot)$  is a non-empty set equipped with a hyperoperation "  $\cdot$  " with the following properties:

- (1)  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ ,  $\forall a, b, c \in P$ ,
- (2)  $\exists! e \in E(P)$  such that  $e \cdot a = a = a \cdot e$ ,  $\forall a \in P$ ,
- (3)  $\forall a \in P \exists! b \in i(a)$  such that  $e \in a \cdot b$ . We denote  $b = a^{-1}$ ,
- (4)  $a \in b \cdot c \Rightarrow b \in a \cdot c^{-1}, c \in b^{-1} \cdot a$ ,  $\forall a, b, c \in P$ .

Let  $X$  be a non-empty subset of a polygroup  $P$  and  $\{A_i \mid i \in J\}$  be the family of all subpolygroups of  $P$  in which contain  $X$ . Then we have

$$\bigcap_{i \in J} A_i = \langle X \rangle = \cup \{x_1^{\varepsilon_1} \cdot \dots \cdot x_k^{\varepsilon_k} \mid x_i \in X, k \in \mathbb{N}, \varepsilon_i \in \{-1, 1\}\}.$$

If  $X = \{x_1, x_2, \dots, x_n\}$ , then the subpolygroup  $\langle X \rangle$  is denoted  $\langle x_1, \dots, x_n \rangle$ . The finitely generated polygroup  $P$  is called *good* if for all  $a, b \in P$ ,  $|ab| < \infty$ .

**Definition 2.6.** [25] *A metric space is an ordered pair  $(X, d)$ , where  $X$  is a set and  $d$  is a metric on  $X$ , i.e., a function  $d: M \times M \rightarrow \mathbb{R}$  such that for all  $x, y, z \in X$ , the followings hold:*

- (1)  $d(x, y) \geq 0$ ,
- (2)  $d(x, y) = 0 \Leftrightarrow x = y$ ,
- (3)  $d(x, y) = d(y, x)$ ,
- (4)  $d(x, y) \leq d(x, z) + d(z, y)$ .

**Definition 2.7.** [25] *Let  $X$  be a metric space. We define distance point  $x \in X$  from set  $A \subseteq X$  to form  $d(x, A) = \inf\{d(x, a) \mid a \in A\}$  and if  $|A| < \infty$ , then  $d(x, A) = \min\{d(x, a) \mid a \in A\}$ .*

In the following we express the definition of a meter in an arbitrary graph. Recall that a graph  $\Gamma$  consists of points called *vertices* and copies of  $[0, 1]$  connecting pairs of vertices called *edges*. Also, interiors of distinct edges are disjoint. Suppose that we assigned to each edge  $e$  of a given connected graph  $\Gamma$  some positive number  $l(e)$ , (its length). Then we can define on  $\Gamma$  a meter, which we describe it now. For each edge  $e$  fix a homeomorphism  $\phi_e : e \rightarrow [0, 1]$  as in the definition of edge. Define the auxiliary function  $\rho$  in the following way. If  $x, y$  belong to the same edge  $e$ , then define  $\rho(x, y) = l(e) \mid \phi_e(x) - \phi_e(y) \mid$ , and otherwise set  $\rho(x, y) = +\infty$ . Finally, set

$$d(x, y) = \inf \sum_{x=x_0, \dots, x_n=y} \rho(x_i, x_{i+1}).$$

$\{x_i\}_{i \in \mathbb{N}}$  as above is usually called *chain* from  $x$  to  $y$ .

**Lemma 2.8.** ([31], Lemma 2.1.1) *In the above definition we can equivalently only take chains  $x = x_0, \dots, x_n = y$  with the additional constraint that  $x_i$  is a vertex for  $i \neq 0, n$ .*

### 3 Generalization of Cayley graph ( $gCay(P, S)$ )

In this section we extend the notion of Cayley graph of a group to a general framework of hyperstructures. A generalized Cayley graph of a finitely generated polygroup has been introduced and investigated.

**Definition 3.1.** *Let  $P = \langle A \rangle$  be a finitely generated polygroup and  $S = A \cup A^{-1}$ . Then the generalized Cayley graph of  $P$  with respect to  $S$  or for simplicity  $gCay(P, S)$  is the metric graph with*

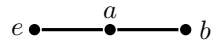
- (1) set of vertices is  $P$ ,
- (2) for  $a, b \in P$  and  $a \neq b$  an edge connecting  $a, b$  if and only if there exist  $s \in S$  such that  $b \in as$  or equivalently  $a^{-1}b \cap S \neq \emptyset$ ,
- (3) all edges are of length 1.

In the following  $P = (P, \cdot)$  will always denote a polygroup generated by the finite set  $A$  and assume  $S = A \cup A^{-1}$ . We also for all  $a, b \in P$  denote  $a \cdot b = ab$  and we denote the metric on  $gCay(P, S)$  as  $d_S$ .

**Example 3.2.** Suppose commutative polygroup  $(P, \cdot)$  is as follows:

$\cdot$	$e$	$a$	$b$
$e$	$e$	$a$	$b$
$a$	$a$	$\{e, b\}$	$\{a, b\}$
$b$	$b$	$\{a, b\}$	$\{e, a\}$

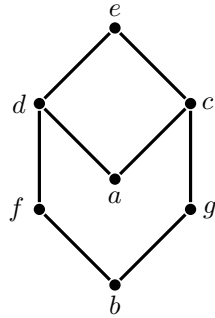
$P$  is generated by  $\{a\}$ . Then  $gCay(P, \{a\})$  looks like this:



**Example 3.3.** Let  $P = \{e, a, b, c, d, f, g\}$ . Consider the polygroup  $(P, \cdot)$ , where  $\cdot$  is defined on  $P$  as follows:

$\cdot$	$e$	$a$	$b$	$c$	$d$	$f$	$g$
$e$	$e$	$a$	$b$	$c$	$d$	$f$	$g$
$a$	$a$	$e$	$b$	$c$	$d$	$f$	$g$
$b$	$b$	$b$	$\{e, a\}$	$g$	$f$	$d$	$c$
$c$	$c$	$c$	$f$	$\{e, a\}$	$g$	$b$	$d$
$d$	$d$	$d$	$g$	$f$	$\{e, a\}$	$c$	$b$
$f$	$f$	$f$	$c$	$d$	$b$	$g$	$\{e, a\}$
$g$	$g$	$g$	$d$	$b$	$c$	$\{e, a\}$	$f$

It is easy to see that  $P = \langle c, d \rangle$ . Set  $S = \{c, d\}$ . Then  $gCay(P, \{c, d\})$  looks like this:



**Lemma 3.4.** *For  $a \neq b$  in  $P$ , we have*

$$d_S(a, b) = \min\{n \mid \exists (s_1, \dots, s_n) \in S^n : b \in as_1 \dots s_n\}.$$

*Proof.* Let  $\min\{n \mid \exists (s_1, \dots, s_n) \in S^n : b \in as_1 \dots s_n\} = t$ . We show that  $d_S(a, b) = t$ . First, let  $x_0 = a, x_1, \dots, x_{n-1}, x_n = b$  be a chain from  $a$  to  $b$  such that by Lemma 2.8,  $x_i \in P$ , for  $0 \leq i \leq n$ . For all  $0 \leq i \leq n-1$ , there exist an edge from  $x_i$  to  $x_{i+1}$ . So by definition of  $gCay(P, S)$  there exist  $(s_1, \dots, s_n) \in S^n$  such that

$$\begin{aligned} x_1 &\in x_0 s_1 = a s_1 \\ x_2 &\in x_1 s_2 \subseteq a s_1 s_2 \\ &\cdot \\ &\cdot \\ x_{n-1} &\in x_{n-2} s_{n-1} \subseteq a s_1 s_2 \dots s_{n-1} \\ b = x_n &\in x_{n-1} s_n \subseteq a s_1 s_2 \dots s_n. \end{aligned}$$

Thus  $t \leq d_S(a, b)$ . Now let  $(s_1, \dots, s_t) \in S^t$  ( $t$ -ary Cartesian product over  $S$ ) such that  $b \in a s_1 s_2 \dots s_t$ . Therefore

$$\begin{aligned} b &\in x_{t-1} s_t & s.t. & x_{t-1} \in a s_1 \dots s_{t-1} \\ x_{t-1} &\in x_{t-2} s_{t-1} & s.t. & x_{t-2} \in a s_1 \dots s_{t-2} \\ &\cdot \\ &\cdot \\ x_3 &\in x_2 s_3 & s.t. & x_2 \in a s_1 s_2 \\ x_2 &\in x_1 s_2 & s.t. & x_1 \in a s_1. \end{aligned}$$

We set  $x_t = b, x_0 = a$ . By definition of  $gCay(P, S)$  there exist an edge from  $x_i$  to  $x_{i+1}$ ,  $i = 0, \dots, t$ . So  $\{x_i\}_{i=0}^t$  is a chain from  $a$  to  $b$ . Therefore  $d_S(a, b) \leq t$ .  $\square$

In other words, for all  $a \neq b$  in  $P$ , the distance between  $a, b$  is the minimum "n" such that  $a^{-1}b \cap s_1 \dots s_n \neq \emptyset$  and  $s_i \in S$ . Notice that  $d_S(a, b) = 0$  if and only if  $a = b$ .

**Remark 3.5.** *For all  $a \in P$  we have  $d_S(e, a) = \min\{n \mid \exists (s_1, \dots, s_n) \in S^n : a \in s_1 \dots s_n\}$ . We denote  $d_S(e, a) = |a|_S$ .*

**Proposition 3.6.** *For all  $a \neq b$  in  $P$ , we have  $d_S(a, b) = \min\{d_S(e, u) \mid u \in a^{-1}b\}$ .*

*Proof.* Let  $d_S(a, b) = t$ . Then there exists  $(s_1, \dots, s_t) \in S^t$  such that  $b \in as_1 \dots s_t$ . So  $b \in au, u \in s_1 \dots s_t$ . Therefore  $u \in a^{-1}b$  and according to Remark 3.5  $d_S(e, u) \leq t$ . Thus  $\min\{d_S(e, u) \mid u \in a^{-1}b\} \leq t$ . Now let  $\min\{d_S(e, u) \mid u \in a^{-1}b\} = k$ . Thus for  $u \in a^{-1}b$  there exist  $(s_1, \dots, s_k) \in S^k$  such that  $u \in s_1 \dots s_k$ . Since  $u \in a^{-1}b$  we have  $b \in au \subseteq as_1 \dots s_k$ . Thus  $d_S(a, b) \leq k$ .  $\square$

**Corollary 3.7.** *Notice that by proposition 3.6, for all  $a \neq b$  in  $P$  we have  $d_S(a, b) = d_S(e, a^{-1}b)$ .*

**Remark 3.8.** *For all  $x, y$  in the  $gCay(P, S)$  that do not lie on a common edge we have*

$$d_S(x, y) = \inf\{d_S(x, a) + d_S(a, b) + d_S(b, y) \mid d_S(x, a) < 1, d_S(b, y) < 1\}.$$

## 4 Hyperaction of $gCay(P, S)$

Using the notion of generalized Cayley graphs of polygroups we introduce the hyperaction of a generalized Cayley graph of a finitely generated polygroup and prove that some hyperactions can be strongly cobounded.

**Definition 4.1.** [36] *Let  $X$  be a non-empty set and  $(H, *)$  be a hypergroup such that  $E(H) \neq \emptyset$ . A left hyperaction of  $H$  on  $X$  is a map  $\cdot: H \times X \rightarrow \mathcal{P}^*(X)$  such that*

- (i) *for all  $a, b \in H$  and for all  $x \in X$ ,  $a \cdot (b \cdot x) = (a * b) \cdot x$ , such that  $A \cdot Y = \cup_{a \in A, y \in Y} a \cdot y$  for all non-empty subsets  $A$  and  $Y$  of  $H$  and  $X$ , respectively;*
- (ii) *for all  $x \in X$  and  $e \in E(H)$ ,  $x \in e \cdot x$ .*

In the following we define a left hyperaction of polygroup  $P$  on  $gCay(P, S)$ . First we need to prove the following lemma.

**Lemma 4.2.** *For all  $a, b, c \in P$ , if there exists an edge from  $b$  to  $c$  then there are edges between  $ab$  and  $ac$ .*

*Proof.* Suppose that there exists an edge from  $b$  to  $c$ . We have  $b^{-1}c \subseteq b^{-1}a^{-1}ac = (ab)^{-1}ac$ . Since  $b^{-1}c \cap S \neq \emptyset$ , so  $(ab)^{-1}ac \cap S \neq \emptyset$ . Therefore there is at least an edge between  $ab$  and  $ac$ .  $\square$

Now for  $a \in P$  and  $x \in gCay(P, S)$  we define the map

$$\varphi: P \times gCay(P, S) \rightarrow \mathcal{P}^*(gCay(P, S))$$

such that  $\varphi(a, x) = a \cdot x = ax$ , if  $x$  is a vertex and for  $x$  on the edge from, say,  $b$  to  $c$ ,

$$\varphi(a, x) = a \cdot x = \{\alpha \in E_{pq} \mid d_S(p, \alpha) = d_S(b, x), p \in ab, q \in ac, p^{-1}q \cap S \neq \emptyset\},$$

where  $E_{pq}$  is edge from  $p$  to  $q$ .

**Theorem 4.3.**  $\varphi$  is a left hyperaction of  $P$  on  $gCay(P, S)$ .

*Proof.* We give a brief proof for showing that  $\varphi$  is a hyperaction. If  $x$  is a vertex then  $\varphi(a, x) = a \cdot x = ax$ . It is clear that the conditions (i) and (ii) in Definition 3.1 hold. So let  $x$  lies on the edge from  $b$  to  $c$ . By Lemma 3.2, there are edges from  $ab$  to  $ac$ . Thus  $\emptyset \neq \varphi(a, x) \subseteq gCay(P, S)$ . Now let  $e \in E(P)$ . We have

$$\begin{aligned} \varphi(e, x) &= e \cdot x = \{\alpha \mid d_S(p, \alpha) = d_S(b, x), p \in eb = b, q \in ec = c, p^{-1}q \cap S \neq \emptyset\} \\ &= \{\alpha \mid d_S(b, \alpha) = d_S(b, x), b^{-1}c \cap S \neq \emptyset\}. \end{aligned}$$

Therefore  $x \in e \cdot x$ . Now let  $a_1, a_2 \in P$ . We show that  $a_1 \cdot (a_2 \cdot x) = (a_1 a_2) \cdot x$ , i.e.,

$$\bigcup_{t \in a_2 \cdot x} a_1 \cdot t = \bigcup_{s \in a_1 a_2} s \cdot x.$$

Let  $u \in \bigcup_{t \in a_2 \cdot x} a_1 \cdot t$ , so  $d_S(p, u) = d_S(v, t) = d_S(b, x), v \in a_2 b, p \in a_1(a_2 b) = (a_1 a_2)b$ . Thus  $p \in sb$  such that  $s \in a_1 a_2$ . Therefore  $u \in \bigcup_{s \in a_1 a_2} s \cdot x$ . Now let  $u \in \bigcup_{s \in a_1 a_2} s \cdot x$ . So  $d_S(p, u) = d_S(b, x)$  such that  $p \in sb \subseteq (a_1 a_2)b = a_1(a_2 b)$ . Therefore  $p \in a_1 v, v \in a_2 b$ . Because  $t \in a_2 \cdot x$ , so  $d_S(p', t) = d_S(b, x)$  in which  $p' \in a_2 b$ . Hence  $d_S(p, u) = d_S(p', t)$ . Therefore  $u \in \bigcup_{t \in a_2 \cdot x} a_1 \cdot t$ .  $\square$

Recall that, if  $(X, d)$  be a metric space, the *open ball* of radius  $r > 0$ , centered at  $x_0 \in X$ , denoted by  $B_r(x_0)$ , is defined by  $B_r(x_0) = \{x \in X \mid d(x, x_0) < r\}$ .

**Definition 4.4.** A hyperaction of the hypergroup  $H$  on the metric space  $X$ , say,  $\cdot : H \times X \rightarrow \mathcal{P}^*(X)$  is proper if for any  $x \in X$  and any ball  $B \subseteq X$ , there are only finitely many elements of  $H$ , say,  $h_i$ , such that  $h_i \cdot x \cap B \neq \emptyset$ .

**Lemma 4.5.** Let  $P$  be a good polygroup. Then for all ball  $B_r(x_0) \subseteq gCay(P, S)$  we have  $|P \cap B_r(x_0)| < \infty$ .

*Proof.* Let  $h_i \in P \cap B_r(x_0), i \in \mathbb{N}$ . So  $h_i$  is a vertex and  $d_S(x_0, h_i) < r$ . We consider two case. In the first case, let  $x_0 \in P$ . If  $r < 1$ , then  $h_i = x_0$ . It is sufficient to prove the problem for  $1 \leq r$ . We have  $d_S(x_0, h_i) < r$ , thus there exist  $n < r$  and  $(s_{i1}, \dots, s_{in}) \in S^n$  such that  $x_0^{-1}h_i \cap s_{i1} \dots s_{in} \neq \emptyset$ . We set

$$C = \cup \{s_{i1} \dots s_{in} \mid x_0^{-1}h_i \cap s_{i1} \dots s_{in} \neq \emptyset, (s_{i1}, \dots, s_{in}) \in S^n, n < r\},$$



Since  $n < r$  and by assumption  $|s_{i_1} \dots s_{i_n}| < \infty$ , so  $C$  is finite (notice that  $P$  is a good polygroup). Therefore there exists  $j$  such that for all  $k \geq j$  we have  $x_0^{-1}h_k \cap C \subseteq \bigcup_{i=1}^{i=j} x_0^{-1}h_i \cap C$ . Now let  $a_i \in x_0^{-1}h_i \cap C$ , thus  $h_i \in x_0 a_i$ . So  $\{h_i\}_{i=1}^{i=j} \subseteq \bigcup_{a_i \in C} x_0 a_i$ . Since  $|C| < \infty$ ,  $|x_0 a_i| < \infty$ , so  $|\{h_i\}_{i=1}^{i=j}| < \infty$ . On the other side  $\{h_s\}_{s \geq j} \subseteq \{h_i\}_{i=1}^{i=j}$ . Hence number of  $h_i$  is finite. In the second case, let  $x_0$  be on the edge from  $a$  to  $b$  and let  $h_i \in P \cap B_r(x_0)$ ,  $i \in \mathbb{N}$ . By triangular inequality we have  $d_S(a, h_i) \leq d_S(a, x_0) + d_S(x_0, h_i) < 1 + r$ . Now according to the first case, number of  $h_i$  is finite.  $\square$

**Theorem 4.6.** *If  $P$  is a good polygroup, then the hyperaction of  $P$  on  $gCay(P, S)$  is proper.*

*Proof.* Consider hyperaction

$$\cdot : P \times gCay(P, S) \rightarrow \mathcal{P}^*(gCay(P, S)).$$

Let the open ball  $B_r(x_0)$  and  $x \in gCay(P, S)$ . If  $x \in P$ , then for all  $h_i \in P$  we have  $h_i \cdot x \subseteq P$ . Now if  $|\{h_i \mid h_i \cdot x \cap B_r(x_0) \neq \emptyset\}|$  is infinite, then  $|P \cap B_r(x_0)| = \infty$  which is a contradiction by Lemma 3.5. Therefore there exist only finitely many  $h_i \in P$  such that  $h_i \cdot x \cap B_r(x_0) \neq \emptyset$ . In other case, let  $x$  be on the edge  $a$  to  $b$ . In this case let  $h_i \in P$  and  $z \in h_i \cdot x \cap B_r(x_0)$ , then  $d_S(z, x_0) < r$  and there exists  $p \in h_i a$  such that  $d_S(p, z) = d_S(a, x)$ . We have  $d_S(p, x_0) \leq d_S(p, z) + d_S(z, x_0) < 1 + r$ . Therefore  $p \in B_{r+1}(x_0)$ . Hence  $h_i a \cap B_{r+1}(x_0) \neq \emptyset$ . By previous part number of  $h_i$  is finite. So the hyperaction is proper.  $\square$

**Definition 4.7.** *A hyperaction of a hypergroup  $H$  on the metric space  $X$ , say,  $\cdot : H \times X \rightarrow \mathcal{P}^*(X)$  is called cobounded if there exists a ball  $B_r(x_0) \subseteq X$  such that for all  $x \in X$ , there is  $h \in H$ , that  $d_X(x, h \cdot x_0) < r$ . Moreover a cobounded hyperaction is said strongly cobounded if, for all  $g, h \in H$  we have*

$$d_X(g \cdot x_0, h \cdot x_0) = d_X(x_0, g^{-1}h \cdot x_0).$$

**Theorem 4.8.** *Let  $P$  be a finitely generated polygroup. Then the hyperaction of  $P$  on  $gCay(P, S)$ , i.e.  $\varphi$ , is strongly cobounded.*

*Proof.* For proving coboundedly, we consider  $x_0 = e$  and  $r = 1$ . Let  $y \in gCay(P, S)$ . If  $y \in P$ , then  $d_S(y, y \cdot e) = d_S(y, y) = 0 < 1$ . If  $y$  is on the edge from  $a$  to  $b$ , then  $d_S(y, a \cdot e) = d_S(y, a) < 1$ . Therefore  $\varphi$  is cobounded. Now by Corollary 3.7, for  $a, b \in P$  we have  $d_S(a \cdot e, b \cdot e) = d_S(a, b) = d_S(e, a^{-1}b) = d_S(e, a^{-1}b \cdot e)$ . Consequently  $\varphi$  is strongly cobounded.  $\square$

## 5 Generalized cayley graphs and quasi-isometries

In this section we show that the  $g$ Cayley graph of a given polygroup is well-defined up to quasi-isometry.

**Definition 5.1.** *Let  $X, Y$  be metric spaces and let  $f : X \rightarrow Y$  be a map from  $X$  to  $Y$ . We say that  $f$  is a  $(K, C)$ -quasi-isometric embedding if, for all  $x, y \in X$  we have*

$$\frac{d_X(x, y)}{K} - C \leq d_Y(f(x), f(y)) \leq Kd_X(x, y) + C.$$

*The  $(K, C)$ -quasi-isometric embedding  $f$  is a  $(K, C)$ -quasi-isometry if, for all  $y \in Y$  there exist some  $x \in X$  with  $d_Y(f(x), y) \leq C$ .*

**Example 5.2.** *Let  $x, y \in \mathbb{R}^2$ , the map  $t \mapsto tx + y$  from  $\mathbb{R}$  to  $\mathbb{R}^2$  is a quasi-isometric embedding.*

**Proposition 5.3.** ([31], Proposition 3.0.3.) *Composition of quasi-isometric embedding (quasi-isometries) is a quasi-isometric embedding (quasi-isometries).*

Now we show that the  $g$ Cayley graph of a given polygroup is well-defined up to quasi-isometry.

**Theorem 5.4.** *Let  $P$  be a polygroup and  $A, A'$  be two finite generating sets for  $P$ . Also let  $S = A \cup A^{-1}$  and  $S' = A' \cup A'^{-1}$ . Then the identity  $id : P \rightarrow P$  extends to a quasi-isometry  $h : gCay(P, S) \rightarrow gCay(P, S')$ .*

*Proof.* Consider the composition

$$gCay(P, S) \xrightarrow{\psi} (P, d_S) \xrightarrow{id} (P, d_{S'}) \xrightarrow{\iota} gCay(P, S'),$$

where  $\iota$  is the inclusion map and

$$\psi(a) = \begin{cases} a & \text{if } a \in P \\ h & \text{if } h \in P, d(a, h) \leq \frac{1}{2} \end{cases} \quad \text{otherwise.}$$

For all  $a \in gCay(P, S)$  we have  $d_S(\psi(a), a) \leq \frac{1}{2}$ , so by triangular inequality

$$\begin{aligned} d_S(a, b) - 1 &\leq d_S(a, \psi(a)) + d_S(\psi(a), \psi(b)) + d_S(\psi(b), b) - 1 \\ &\leq d_S(\psi(a), \psi(b)) \leq d_S(\psi(a), a) + d_S(a, b) + d_S(b, \psi(b)) \\ &\leq d_S(a, b) + 1. \end{aligned}$$

Therefore  $\psi(a)$  is  $(1, 1)$ -quasi-isometry. Also  $\iota$  is  $(1, 1)$ -quasi-isometry. By Proposition 5.3, the above composition is a quasi-isometry if  $id : (P, d_S) \rightarrow (P, d_{S'})$  is too. We set

$$M = \text{Max}\{|x'|_S, |x|_{S'} : x \in S, x' \in S'\}.$$

Now let  $a, b \in P$  and  $d_S(a, b) = k$ . By Remark 3.5 and Proposition 3.6 we have

$$|a^{-1}b|_S = d_S(e, a^{-1}b) = \min\{d_S(e, u) \mid u \in a^{-1}b\} = d_S(a, b) = k.$$

Hence there exists  $(s_1, \dots, s_k) \in S^k$  such that  $a^{-1}b \cap s_1 \dots s_k \neq \emptyset$ . On the other side we let

$$\begin{array}{l} |s_1|_{S'} = m_1 \Rightarrow \exists (s'_{1,1}, \dots, s'_{1,m_1}) \in S'^{m_1} : s_1 \in s'_{1,1} \dots s'_{1,m_1} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ |s_k|_{S'} = m_k \Rightarrow \exists (s'_{k,1}, \dots, s'_{k,m_k}) \in S'^{m_k} : s_k \in s'_{k,1} \dots s'_{k,m_k}. \end{array}$$

For some  $m_i \leq M$ . Therefore

$$\emptyset \neq a^{-1}b \cap \prod_{i=1}^k s_i \subseteq a^{-1}b \cap \left( \prod_{i=1}^{m_1} s'_{1,i} \right) \dots \left( \prod_{i=1}^{m_k} s'_{k,i} \right).$$

Hence

$$d_{S'}(a, b) = |a^{-1}b|_{S'} \leq Mk \leq M d_S(a, b).$$

The inequality  $d_S \leq M d_{S'}$  follows using the same argument. Consequently

$$\frac{d_{S'}(a, b)}{M} \leq d_S(id(a), id(b)) \leq M d_{S'}(a, b).$$

Therefore  $id$  is a quasi-isometry embedding.

Thus  $id : gCay(P, S) \rightarrow gCay(P, S')$  is a quasi-isometry.  $\square$

## 6 Generalized Cayley graphs and geodesic metric spaces

In this section, we express the definition of geodesic metric space and then we show that  $gCay(P, S)$  is a geodesic metric space. Also we prove that polygroup  $P$  such that for all  $a, b \in P$ ,  $|ab| < \infty$  acts properly and strongly coboundedly on the geodesic metric space  $X$ , then  $P$  is finitely generated and  $gCay(P, S)$  is quasi-hyperisometric to  $X$ .

If  $X$  is a metric space then a *path* in  $X$  is a continuous map  $\alpha : [0, 1] \rightarrow X$ . We define the *length* of  $\alpha$  as

$$l(\alpha) = \sup \sum_{0=t_0 \leq \dots \leq t_n=1} d_X(\alpha(t_i), \alpha(t_{i+1})),$$

where  $\alpha(0)$  is initial point and  $\alpha(1)$  is terminal point.

**Proposition 6.1.** ([31], Remark 2.4.1) *For any path  $\alpha$ , we have*

$$l(\alpha) \geq d_X(\alpha(0), \alpha(1)).$$

**Proposition 6.2.** ([31], Remark 2.4.2) *If we denote the concatenation of the paths  $\alpha, \beta$  by  $\alpha * \beta$ , then  $l(\alpha * \beta) = l(\alpha) + l(\beta)$ .*

In the following we define the notion of geodesic metric space.

**Definition 6.3.** *A path  $\alpha$  is geodesic if  $l(\alpha) = d_X(\alpha(0), \alpha(1))$ . The metric space  $X$  is geodesic if for all pair of points of  $X$  there is a geodesic connecting them.*

**Theorem 6.4.** *The metric space  $gCay(P, S)$  is a geodesic.*

*Proof.* Let  $x, y \in gCay(P, S)$ . We consider three cases. First, if  $x$  and  $y$  lie on a common edge, then path  $\alpha$  is partial of edge that placed among  $x$  and  $y$ . So it is clear that  $l(\alpha) = d_S(x, y) = d_S(\alpha(0), \alpha(1))$ . Second, if  $x$  and  $y$  be vertex and  $d_S(x, y) = n$ , then by Lemma 3.4 there exist  $(s_1, \dots, s_n) \in S^n$  such that  $y \in xs_1 \dots s_n$ . We consider  $x_i \in xs_1 \dots s_i$ , where  $1 \leq i \leq n$  and  $x_0 = x, x_n = y$ . Also  $d_S(x_i, x_{i+1}) = 1$ , for  $0 \leq i \leq n-1$ . In this case let path  $\alpha_i : [0, 1] \rightarrow gCay(P, S)$  be edge from  $x_i$  to  $x_{i+1}$  such that  $\alpha_i(0) = x_i$  and  $\alpha_i(1) = x_{i+1}$  for  $0 \leq i \leq n-1$ . We have  $l(\alpha_i) = 1$ . If we set  $\alpha = \alpha_0 * \alpha_1 * \dots * \alpha_{n-1}$ , then by Proposition 6.2  $l(\alpha) = l(\alpha_0) + \dots + l(\alpha_{n-1}) = n = d(x, y)$ . Third, if  $x, y \notin P$  and  $x, y$  do not lie on a common edge, then according to first and second cases also Remark 3.8, the proof is obviously.  $\square$

**Definition 6.5.** *Let  $X, Y$  be metric spaces and let  $f : X \rightarrow \mathcal{P}^*(Y)$  be a map from  $X$  to  $\mathcal{P}^*(Y)$ . We say that  $f$  is a  $(K, C)$ -quasi-hyperisometric embedding if, for all  $x, y \in X$  we have*

$$\frac{d_X(x, y)}{K} - C \leq d_Y(f(x), f(y)) \leq Kd_X(x, y) + C.$$

*The  $(K, C)$ -quasi-hyperisometric embedding  $f$  is a  $(K, C)$ -quasi-hyperisometry if, for all  $y \in Y$  there exist some  $x \in X$  with  $d_Y(f(x), y) \leq C$ .*

**Theorem 6.6.** *Let  $P$  be a polygroup with this condition that for all  $a, b \in P$ ,  $|ab| < \infty$ . If  $P$  acts properly and strongly cobounded on the geodesic metric space  $X$ , then*

- (1)  $P$  is a good polygroup,
- (2)  $gCay(P, S)$  is quasi-hyperisometric to  $X$ .

*Proof.* (1) Consider the proper and strongly cobounded hyperaction  $\cdot : P \times X \rightarrow \mathcal{P}^*(X)$ . Since hyperaction is cobounded there exists  $B_r(x_0)$  such that for all  $x \in X$ , there exists  $c \in P$ , that  $d_X(x, c \cdot x_0) < r$ . Let  $a \in P$  and  $b \in a \cdot x_0$ . We can connect  $x_0$  to  $b$  by a geodesic. On this geodesic we consider a sequence of points  $x_0 = y_0, y_1, \dots, y_n = b$  such that  $y_i \in X, d_X(y_0, y_1) \leq 1, d_X(y_{n-1}, y_n) \leq 1$  and  $d_X(y_i, y_{i+1}) = 1$ , for  $1 \leq i \leq n-2$ . By coboundedly for all  $0 \leq i \leq n$ , there exist  $a_i \in P$  such that  $d_X(y_i, a_i \cdot x_0) < r$ , where  $a_0 = e$  and  $a_n = a$ . We set

$$S = \{h \in P \mid h \cdot x_0 \cap B_{2r+1}(x_0) \neq \emptyset\}.$$

Notice that  $\emptyset \neq S$  is finite, because the hyperaction is proper. Now we show that for  $a_i \in P, 0 \leq i \leq n$  we have  $a_i^{-1}a_{i+1} \cap S \neq \emptyset$ . By strongly coboundedly for all  $i, 0 \leq i \leq n$ , we have

$$\begin{aligned} d_X(x_0, a_i^{-1}a_{i+1} \cdot x_0) &= d_X(a_i \cdot x_0, a_{i+1} \cdot x_0) \\ &\leq d_X(a_i \cdot x_0, y_i) + d_X(y_i, y_{i+1}) + d_X(y_{i+1}, a_{i+1} \cdot x_0) \\ &\leq 2r + 1. \end{aligned}$$

Therefore  $a_i^{-1}a_{i+1} \cap S \neq \emptyset$ . Thus there is  $s_{i+1} \in S$  such that  $a_{i+1} \in a_i s_{i+1}$ , for  $0 \leq i \leq n-1$ . So we have

$$a = a_n \in a_{n-1}s_n \subseteq a_{n-2}s_{n-1}s_n \subseteq \dots \subseteq a_0s_1s_2\dots s_n = s_1s_2\dots s_n,$$

i.e. we write an arbitrary  $a \in P$  as a product of elements of  $S$ . Hence polygroup  $P$  generates by finite set  $S$  and so it is a good polygroup. (2) We define the map  $f : gCay(P, S) \rightarrow \mathcal{P}^*(X)$  by  $f(a) = a \cdot x_0$  for any given choice of  $x_0 \in X$ . Recall that according to Theorem 5.4,  $f$  is defined only on the vertex set. Again we consider geodesic from  $x_0$  to  $b \in a \cdot x_0$ . For all  $b \in a \cdot x_0$  we have  $d_X(x_0, b) \geq n-2$  and thus  $d_X(x_0, a \cdot x_0) \geq n-2$ . Also since  $a \in s_1\dots s_n$ , so

$$|a|_S \leq n \leq d_X(x_0, a \cdot x_0) + 2.$$

Now let  $|a|_S = k$ , thus  $a \in s_1\dots s_k$  such that  $s_i \in S$ . Therefore  $a \in a_{k-1}s_k$ , that  $a_{k-1} \in s_1\dots s_{k-1}$ . By continuing this process we have  $a_{i+1} \in a_i s_{i+1}$  such

that  $a_i \in s_1 \dots s_i$ , for  $1 \leq i \leq k-1$ ,  $a_0 = e$  and  $a_k = a$ . So  $s_{i+1} \in a_i^{-1} a_{i+1}$ . By triangular inequality and strongly cobounded we have

$$\begin{aligned} d_X(x_0, a \cdot x_0) &\leq \sum_{i=0}^{k-1} d_X(a_i \cdot x_0, a_{i+1} \cdot x_0) = \sum_{i=0}^{k-1} d_X(x_0, a_i^{-1} a_{i+1} \cdot x_0) \\ &\leq \sum_{i=0}^{k-1} d_X(x_0, s_{i+1} \cdot x_0) \leq k(2r+1) = |a|_S(2r+1). \end{aligned}$$

Therefore we get

$$d_S(e, a) - 2 \leq d_X(f(e), f(a)) \leq d_S(e, a)(2r+1).$$

In a way similar to Corollary 3.7,  $f$  is a quasi-hyperisometric embedding. Also by coboundedness  $f$  is a quasi-hyperisometry.  $\square$

## 7 Conclusions

Since 1934, when Marty [29] defined hypergroup as a generalization of a group, many connections between hyperstructures and other branches of mathematics have been developed. This theory is rich in applications, for instance in geometry and graphs. Till now the study of Geometric group theory has been devoted to the study of finitely generated groups via exploring the connections between algebraic properties of such groups and topological and geometric properties of spaces on which these groups act. In this article we extend it to the general framework of hyperstructures called polygroups. A generalized Cayley graph of a finitely generated polygroup has been introduced and studied. Based on this notion we have shown that the generalized Cayley graph of a polygroup by two different generators are quasi-isometric. Moreover, a connection between geodesic metric spaces and the generalized Cayley graphs of a polygroup has been investigated.

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